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The Moduli Space and Monodromies of the $N = 2$ Supersymmetric Yang-Mills Theory with any Lie Gauge Groups

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Abstract

We propose a unified scheme for finding the hyperelliptic curve of $N = 2$ SUSY YM theory with any Lie gauge groups. Our general scheme gives the well known results for classical gauge groups and exceptional G_2 group. In particular, we present the curve for the exceptional gauge groups $F_4, E_{6,7,8}$ and check consistency condition for them. The exact monodromies and the dyon spectrum of these theories are determined. We note that for any Lie gauge groups, the exact monodromies could be obtained only from the Cartan matrix.

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1 Introduction

During the two years ago, exact results on the four dimensional quantum field theory have been obtained . Although all of these models are supersymmetric, the study of them, helps to understanding some results on the non-perturbative quantum field theory in four dimensions. The key point in $N = 2$ SUSY YM models was the discovery of a hyperelliptic curve which gives all the informations about low energy effective action. At first, Seiberg and Witten have noticed to this point for $N = 2$ $SU(2)$ SUSY YM theory [1].

This work have been generalized for $SU(N)$ [2], $SO(2n + 1)$ [3], $SO(2n)$ [4], $SP(2n)$ [5] and G_2 [6] gauge groups. Also $N = 2$ supersymmetric gauge theory with matter multiplet in the fundamental representation of the gauge group has been considered in [7]. The low energy effective action for the $N = 2$ gauge theory, written in terms of the $N = 1$ fields is:

$$\frac{1}{4\pi} Im\left(\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A^i}\bar{A}^i + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}(A)}{\partial A^i \partial A^j} W^i_\alpha W^j_\alpha\right), \tag{1.1}$$

where the A^i's are the $N = 1$ chiral field multiplets and W^i_α are the vector multiplets all in the adjoint representation. We note first that the whole theory is described by a single holomorphic function, the prepotential $\mathcal{F}(A)$ which at the classical level is $\mathcal{F} = \frac{1}{2}\tau A^2$,$\tau = i\frac{4\pi}{g^2} + \frac{\theta}{2\pi}$. Second, this theory has a classical potential for scalar fields given by $V(\phi) = tr[\phi, \phi^\dagger]^2$.

The v.e.v. of the scalar componet ϕ of A, determines the moduli space of the Coulomb phase of the theory, which is in turn parametrized by the invariants of the gauge group.

$$<\phi>=\sum a_i H_i, \tag{1.2}$$

with H_i's denoting the Cartan generators.

In a generic point of the moduli space, the gauge group will be broken to $U(1)^r$, where r is the rank of the group and the W-bosons corresponding to the roots $\vec{\alpha}$ of the gauge group become massive with a mass proportional to $(\vec{a}\cdot\vec{\alpha})^2$. When the v.e.v of scalar field is orthogonal to a root, the corresponding W-boson becomes massless and the symmetry is enhanced, therefore the low energy effective action is not valid. In the quantum case, the theory is described by a scale factor Λ , and the prepotential in the limit of weak coupling gets one loop correction

$$\mathcal{F} = \frac{i}{4\pi}\sum_{\vec{\alpha}\in\delta^+}(\vec{\alpha}\cdot\vec{a})^2 \ln\frac{(\vec{\alpha}\cdot\vec{a})^2}{\Lambda^2}, \tag{1.3}$$

where δ^+ is the set of positive roots of the gauge group. If the gauge group is a non-simply laced group, then in the (1.3), $(\vec{\alpha}\cdot\vec{a})^2$ must be replaced by $\frac{2(\vec{\alpha}\cdot\vec{a})^2}{(\vec{\alpha}\cdot\vec{\alpha})}$ [8]. One can see that the logarithmic term diverges where $\vec{a}\cdot\vec{\alpha} = 0$, which are exactly the walls of the Weyl chambers in complexified Cartan subalgebra . The prepotential is

not a single valued function of $\vec{a}$, therefore encircling the singularities in the moduli space gives non-trivial monodromy. One can consider the locus of the singularities to be the zeros of the Weyl invariant classical discriminant defined by

$$\Delta_{cl} = \prod_{\vec{\alpha}\in\delta^+} (\vec{a}\cdot\vec{\alpha})^2. \tag{1.4}$$

All the semiclassical monodromies are generated by r simple monodromies corresponding to the r simple roots of the gauge group. The simple monodromy of the simple root α_i is [3]

$$M^{(r_i)} = \begin{pmatrix} (r_i^{-1})^t & \alpha_i \otimes \alpha_i \\ 0 & r_i \end{pmatrix}, \tag{1.5}$$

where r_i is the Weyl reflection corresponding to the root α_i, and acts on the pair $\begin{pmatrix} \vec{a}^D \\ \vec{a} \end{pmatrix}$ where $a_i^D = \frac{\partial \mathcal{F}}{\partial a_i}$. Note that the off diagonal terms in (1.5) can be changed by a change of homology basis ($a_i^D \to a_i^D + K_{ij}a_j$). One can also decompose (1.5) to a classical part that is generated by Weyl reflections and a quantum part $M^{(r_i)} = P^{(r_i)}T_i^{-1}$ [2]. The important discovery in the $N = 2$ gauge theories has been the realization that the prepotentials can be described with the aid of a family of complex curves, with the identification of the v.e.v., a_i and its dual a_i^D, with the periods of the curve ,

$$a_i = \oint_{\alpha_i} \lambda \quad \text{and} \quad a_i^D = \oint_{\beta_i} \lambda, \tag{1.6}$$

where α_i and β_i are the homology cycles of the corresponding Riemann surface.

The curves and the Riemann surfaces have been found for the A_n, B_n, C_n, D_n and G_2 Lie groups [1, 2, 3, 4, 5, 6]. Especially, in [9], the authors claimed a unified framework for finding the curves of $N = 2$ SUSY YM theory for all gauge groups. However they do not give any explicit form of curves for the exceptional gauge groups and any physical checks on the curves.

In this article, we present a unified scheme for finding the hyperelliptic curve of the $N = 2$ SUSY YM theory with any Lie gauge group. The paper is organised as follows:

In section two, we introduce our unified scheme and redrive previously obtained results for classical Lie gauge groups and G_2.

In section three, we present the curves for exceptional Lie groups $F_4, E_{6,7,8}$ and check their physical consistency.

2 Unified Scheme

We will now construct a complex curve for any gauge group in the form

$$y^2 = W^2(x) - \Lambda^{2\tilde{h}} x^k \tag{2.1}$$

The power of Λ which is twice the dual Coxeter number $\tilde{h}$, is determined by $U(1)_R$ anomaly [10], and k is determined by the degree of $W(x)$, the classical curve of gauge group. The singularity structure of classical theory must be encoded in $W(x)$, which is determined by Weyl group and its discriminant vanishes on the walls of the Weyl chamber. Proceeding in this manner, and requiring $W(x)$ should be Weyl invariant, we propose

$$W(x) = \prod_i (x - \vec{\lambda}_i \cdot \vec{a})^2, \tag{2.2}$$

where $\vec{\lambda}_i$ are non-zero weight charge vectors of the fundamental representation of the gauge group with the least dimension. The discriminant of (2.2) is

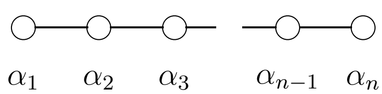
$$\Delta_W = \prod_{i<j} \{(\vec{\lambda}_i - \vec{\lambda}_j) \cdot \vec{a}\}^2, \tag{2.3}$$

and we have moduli space singularities when $\Delta_W = 0$.

Now, let us reproduce the results of classical gauge group [1, 2, 3, 4, 5] and G_2 [6].

I. A_n Series

We take the following Dynkin diagram for $SU(n+1)$



We choose the fundamental representation Λ_1 which its dimension is $n+1$, so (2.2) is

$$W(x) = (x - a_1)(x - (a_2 - a_1)) \cdots (x - (a_n - a_{n-1}))(x + a_n), \tag{2.4}$$

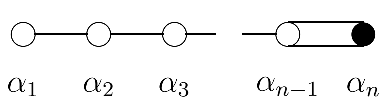
and the quantum curve (2.1) is

$$y^2 = W^2(x) - \Lambda^{2(n+1)}, \tag{2.5}$$

which is the same curve as that obtained in [2].

II. B_n Series

The Dynkin diagram of $SO(2n+1)$ is



We choose the fundamental representation Λ_1 which its dimension is $2n+1$, so (2.2) is

$$W(x) = (x^2 - a_1^2)(x^2 - (a_2 - a_1)^2) \cdots (x^2 - (a_n - a_{n-1})^2), \tag{2.6}$$

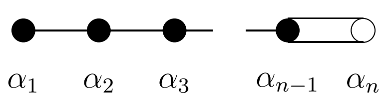
and the quantum curve (2.1) is

$$y^2 = W^2(x) - \Lambda^{(4n-2)} x^2. \tag{2.7}$$

It can easily be seen that by changing the basis of the root space to orthogonal basis, (2.7) is exactly the same curve as that obtained in [3].

III. C_n **Series**

The Dynkin diagram for $SP(2n)$ group is



We choose the fundamental representation Λ_1 which its dimension is $2n$, so (2.2) is

$$W(x) = (x^2 - a_1^2)(x^2 - (a_2 - a_1)^2) \cdots (x^2 - (2a_n - a_{n-1})^2), \tag{2.8}$$

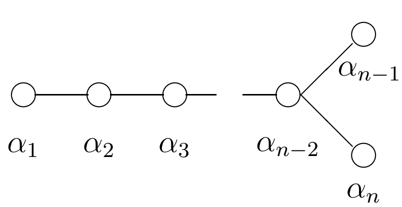
and the quantum curve (2.1) is

$$y^2 = W^2(x) - \Lambda^{2(n+1)} x^{2n-2}, \tag{2.9}$$

which is the double cover of the curve obtained in [5].

IV. D_n **Series**

The Dynkin diagram of $SO(2n)$ is



We choose the fundamental representation Λ_1 which its dimension is $2n$, so (2.2) is

$$\begin{aligned} W(x) &= (x^2 - a_1^2)(x^2 - (a_2 - a_1)^2) \cdots (x^2 - (a_{n-2} - a_{n-3})^2) \\ &\quad (x^2 - (a_n + a_{n-1} - a_{n-2}))(x^2 - (a_n - a_{n-1})^2), \end{aligned} \tag{2.10}$$

and the quantum curve (2.1) is

$$y^2 = W^2(x) - \Lambda^{4(n-1)} x^4, \tag{2.11}$$

which in a suitable orthogonal basis is exactly equivalent to the curve given in [4].

As we shall see, in contrast to the previous cases, there are additional "singularities", we shall encounter the same situation for exceptional gauge groups. The discriminant of classical curve (2.10) has the following decomposition

$$\Delta_W = t^2 \Delta, \tag{2.12}$$

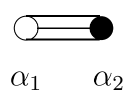
in the notation of [4], where $t^2 = \prod_{i=1}^{2n} \vec{\lambda}_i \cdot \vec{a}$. The quantum discriminant has also a similar decomposition

$$\Delta_\Lambda = t^4 \Delta^+ \Delta^-. \tag{2.13}$$

The apparent "singularity" at $t = 0$ is not a physical singularity which means there is no massless dyon corresponding to it, i.e. the monodromy corresponding to encircling $t = 0$, in the quantum moduli space, is trivial. [4, 5].

V. G_2 Case

The Dynkin diagram is



We choose the fundamental representation Λ_2 which its dimension is 7, so (2.2) is

$$W(x) = (x^2 - a_2^2)(x^2 - (a_2 - a_1)^2)(x^2 - (a_1 - 2a_2)^2), \tag{2.14}$$

and the quantum curve (2.1) is

$$y^2 = W^2(x) - \Lambda^8 x^4, \tag{2.15}$$

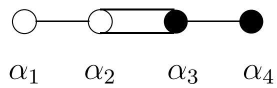
which is the same as the curve given in [6].

3 Exceptional Lie Gauge Groups

In this section, we apply our scheme to exceptional Lie gauge groups $F_4, E_{6,7,8}$. In each cases, we present classical and quantum curves, simple semiclassical and exact monodromies and check that the simple semiclassical monodromies can be obtained by the product of a pair of simple exact monodromies. We also obtain quantum shift matrix for each case.

I. F_4 Gauge Group

The group F_4 is of the rank 4 and dimension 52. The Dynkin diagram of F_4 is



The Weyl group of F_4 is the group of order 1152 which is generated by

$$\begin{aligned} r_1 &: (a_1, a_2, a_3, a_4) \to (-a_1 + a_2, a_2, a_3, a_4), \\ r_2 &: (a_1, a_2, a_3, a_4) \to (a_1, a_1 - a_2 + 2a_3, a_3, a_4), \\ r_3 &: (a_1, a_2, a_3, a_4) \to (a_1, a_2, a_2 - a_3 + a_4, a_4), \\ r_4 &: (a_1, a_2, a_3, a_4) \to (a_1, a_2, a_3, a_3 - a_4). \end{aligned} \tag{3.1}$$

We choose the fundamental representation Λ_4 which its dimension is 26, so (2.2) is

$$\begin{aligned} W \quad (x) &= (x^2 - a_4^2)(x^2 - (a_4 - a_3)^2)(x^2 - (a_3 - a_2)^2) \\ &(x^2 - (a_3 - a_2 + a_1)^2)(x^2 - (a_3 - a_1)^2)(x^2 - (a_4 - a_3 + a_1)^2) \\ &(x^2 - (a_4 - a_1)^2)(x^2 - (a_4 - a_3 + a_2 - a_1)^2)(x^2 - (a_4 - a_2 + a_1)^2) \\ &(x^2 - (a_4 + a_3 - a_2)^2)(x^2 - (a_4 - 2a_3 + a_2)^2)(x^2 - (2a_4 - a_3)^2), \end{aligned} \tag{3.2}$$

and the quantum curve (2.1) is

$$y^2 = W^2(x) - \Lambda^{18} x^{30}. \tag{3.3}$$

By our construction, the curve (3.3) is Weyl invariant and hence could be expressed in terms of Casimir invariants of the F_4, that are u_2, u_6, u_8, u_{12}. The classical discriminant of (3.2) is factorized in the following form

$$\Delta_W = f \prod_{\vec{\alpha} \in \delta^+(F_4)} (\vec{a} \cdot \vec{\alpha}),^2 \tag{3.4}$$

where f is a Weyl invariant function of $\vec{a}$. As the D_n case, there are unexpected "singularities" where $f = 0$. We shall discuss them shortly. The quantum discriminant is

$$\Delta_\Lambda = \prod_{i<j} (e_i^+ - e_j^+)^2 (e_i^- - e_j^-)^2, \tag{3.5}$$

and can be factorized as follows

$$\Delta_\Lambda = f^+ f^- \Delta^+ \Delta^-. \tag{3.6}$$

The hyperelleptic curve has 24 cuts, and the homology cycles around them are

$$\begin{array}{lll} \gamma_1 = \alpha_4, & \gamma_2 = \alpha_4 - \alpha_3, & \gamma_3 = \alpha_3 - \alpha_2, \\ \gamma_4 = \alpha_3 - \alpha_1, & \gamma_5 = \alpha_4 - \alpha_1, & \gamma_6 = 2\alpha_4 - \alpha_3, \\ \gamma_7 = \alpha_4 + 2\alpha_3 - \alpha_2, & \gamma_8 = \alpha_3 - \alpha_2 + \alpha_1, & \gamma_9 = \alpha_4 + \alpha_3 - \alpha_2, \\ \gamma_{10} = \alpha_4 - \alpha_2 + \alpha_1, & \gamma_{11} = \alpha_4 - \alpha_3 + \alpha_1, & \gamma_{12} = \alpha_4 - \alpha_3 + \alpha_2 - \alpha_1, \end{array} \tag{3.7}$$

and $\gamma_{12+i}, i = 1, \cdots, 12$ are related to γ_i by parity $x \to -x$. The intersection requirements of the cycles β_i with α_i, determine the β_i to be

$$\gamma_1^D = \beta_1 + \beta_2 + \beta_3 + \beta_4, \quad \gamma_2^D = -\beta_1 - \beta_2 - \beta_3, \quad \gamma_3^D = -\beta_2, \quad \gamma_4^D = -\beta_1, \tag{3.8}$$

where γ_i^D's are the conjugate cycles to γ_i. As F_4 is non-simply laced group, we must use the modified Picard-Lefschetz formula for obtaining the exact monodromies [3, 4]. In the quantum moduli space, γ_i's do not vanish anywhere as far as the semiclassical monodromies are concerned, unless $\Lambda = 0$. For $\Lambda \to 0$, the γ_i cycles vanish, since $e_i^+ \to e_i^-$. Then it is straightforward to compute the monodromies for the singularities at $\Lambda \to 0$, which we denote by B_i, $i = 1, \cdots, 12$ corresponding to each γ_i. By multiplying these matrices, one can obtain the quantum shift matrix, $\prod_{i=1}^{12} B_i = T^{-3}$, where $T = \begin{pmatrix} \mathbb{1} & C \\ 0 & \mathbb{1} \end{pmatrix}$, and

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}, \tag{3.9}$$

in agreement with the shift obtained from the one loop corrected prepotential under $\Lambda^{18} \to e^{2\pi i\theta}\Lambda^{18}$, $\theta \in [0, 1]$. The simple semiclassical monodromies obtained from the one loop corrected prepotential are

$$M^{(r_i)} = \begin{pmatrix} (r_i^{-1})^t & -C_i \\ 0 & r_i \end{pmatrix}, \tag{3.10}$$

where

$$\begin{aligned} C_1 &= \begin{pmatrix} -4 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1 & 2 & -2 & 0 \\ 2 & -4 & 4 & 0 \\ -2 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ C_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & -4 & 2 \\ 0 & -1 & 2 & -1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 2 & -4 \end{pmatrix}. \end{aligned} \tag{3.11}$$

To calculate exact monodromies, we look at the vanishing cycles of Riemann surface where Δ^+ or $\Delta^- = 0$. We have chosen a slice in moduli space at fixed u_4, u_6, u_{12} and varying u_2. Fixing a basis in this slice and encircling the singularity, we obtain the following simple monodromies

$$\begin{aligned} M_1 &= M_{(1,0,0,0;0,0,0,0)}, \quad M_2 = M_{(1,0,0,0;-2,1,0,0)}, \\ M_3 &= M_{(0,1,0,0;0,0,0,0)}, \quad M_4 = M_{(0,1,0,0;-1,2,-2,0)}, \\ M_5 &= M_{(0,0,1,0;0,0,0,0)}, \quad M_6 = M_{(0,0,1,0;0,-1,2,-1)}, \\ M_7 &= M_{(0,0,0,1;0,0,0,0)}, \quad M_8 = M_{(0,0,0,1;0,0,-1,2)}, \end{aligned} \tag{3.12}$$

where we use the usual notation

$$M_{(\boldsymbol{g},\boldsymbol{q})} = \begin{pmatrix} \mathbb{1} + \boldsymbol{q}\otimes\boldsymbol{g} & \boldsymbol{q}\otimes\boldsymbol{q} \\ -\boldsymbol{g}\otimes\boldsymbol{g} & \mathbb{1} - \boldsymbol{g}\otimes\boldsymbol{q} \end{pmatrix}. \tag{3.13}$$

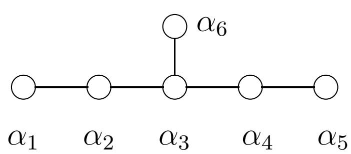
The other monodromies can be obtained from the above simple monodromies by the conjugation. The product of two strong monodromies in (3.12) generates weak monodromies (3.11) by

$$M_1 M_2 = M^{(r_1)}, M_3 M_4 = M^{(r_2)}, M_5 M_6 = M^{(r_3)}, M_7 M_8 = M^{(r_4)}. \tag{3.14}$$

As we noted above, when in the classical discriminant (3.4) $f = 0$ or in the quantum discriminant (3.5) $f^+ = f^- = 0$, there are some "singularities". We argue here that at these "singularities" no BPS saturated states become massless. In fact, by encircling these "singularities", there is no change in the logarithmic term of prepotential, hence there is not any quantum shift for these classical monodromies. Therefore, it seems that in these "singularities", no BPS saturated states become massless. Moreover, by tracing the branch points of some of these "singularities" for some special slices in moduli space, we have checked that their paths do not intersect any other path singularities and hence the monodromies associated with these "singularities" are trivial.

II. E_6 Gauge Group

The group E_6 is of rank 6 and dimension 78 and its Dynkin diagram is



The Weyl group of E_6 is of the order 51840, which is generated by

$$\begin{aligned}
r_1 &: (a_1,a_2,a_3,a_4,a_5,a_6) \to (-a_1+a_2,a_2,a_3,a_4,a_5,a_6),\\
r_2 &: (a_1,a_2,a_3,a_4,a_5,a_6) \to (a_1,a_1-a_2+a_3,a_3,a_4,a_5,a_6),\\
r_3 &: (a_1,a_2,a_3,a_4,a_5,a_6) \to (a_1,a_2,a_2-a_3+a_4+a_6,a_4,a_5,a_6),\\
r_4 &: (a_1,a_2,a_3,a_4,a_5,a_6) \to (a_1,a_2,a_3,a_3-a_4+a_5,a_5,a_6),\\
r_5 &: (a_1,a_2,a_3,a_4,a_5,a_6) \to (a_1,a_2,a_3,a_4,a_4-a_5,a_6),\\
r_6 &: (a_1,a_2,a_3,a_4,a_5,a_6) \to (a_1,a_2,a_3,a_4,a_5,a_3-a_6).
\end{aligned} \tag{3.15}$$

We choose the fundamental representation Λ_1 which its dimension is 27, so (2.2) is

$$\begin{aligned}
W \quad & (x) = (x-a_1)(x-a_2+a_1)(x-a_5+a_2-a_1)(x-a_6-a_4+a_3)\\
& (x-a_6-a_5+a_4)(x+a_6-a_4)(x-a_6+a_5)(x+a_6-a_5+a_4-a_3)\\
& (x+a_6+a_5-a_3)(x-a_5+a_3-a_2)(x+a_5-a_4+a_3-a_2)(x-a_3+a_2)\\
& (x+a_4-a_2)(x+a_5-a_4+a_2-a_1)(x-a_5+a_1)(x+a_4-a_3+a_2-a_1)
\end{aligned}$$

$$
\begin{aligned}
&(x+a_5-a_4+a_1)(x-a_6+a_3-a_1)(x+a_4-a_3+a_1)(x+a_6-a_1)\\
&(x-a_6+a_3-a_2+a_1)(x+a_6-a_2+a_1)(x-a_6+a_2)(x+a_6-a_3+a_2)\\
&(x-a_4+a_3)(x-a_5+a_4)(x+a_5), \qquad (3.16)
\end{aligned}
$$

and the quantum curve (2.1) is

$$
y^2 = W^2(x) - \Lambda^{24} x^{30}. \qquad (3.17)
$$

We can also express the above Weyl invariant curve in terms of the Casimir invariants of the group which are $u_2, u_5, u_6, u_8, u_9, u_{12}$. The classical and quantum discriminants can be factorized as the case of F_4 and hence we have a large number of unphysical "singularities". In this case, like the F_4 case, these "singularities" can not produce any non-trivial monodromies, i.e. massless BPS saturated states. The hyperelliptic curve (3.17) has 27 cuts that can easily be written from classical level surface [11]. These 27 cycles vanish when $\Lambda \to 0$, so one can compute the monodromies for these singularities (B_i, $i = 1, \cdots, 27$) and obtain the quantum shift matrix $\prod_{i=1}^{27} B_i = T^{-6}$ where $T = \begin{pmatrix} \mathbb{1} & C \\ 0 & \mathbb{1} \end{pmatrix}$ and C is the Cartan matrix of E_6, as we expected from one loop corrected prepotential. The simple semiclassical monodromies obtained from the one loop corrected prepotential are in the form (3.10). Now the C_i's are

$$
C_1 = \begin{pmatrix} -4 & 2 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad
C_2 = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 & 0 \\ 2 & -4 & 2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 2 & -4 & 2 & 0 & 2 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \end{pmatrix}, \quad
C_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -4 & 2 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
C_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad
C_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -4 \end{pmatrix}. \qquad (3.18)
$$

Finally, to obtain the exact monodromies for E_6 group, we choose a slice in moduli space at fixed $u_5, u_6, u_8, u_9, u_{12}$ and varying u_2, then we get the simple exact

monodromies

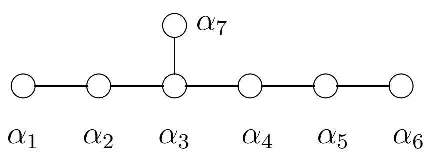
$$
\begin{array}{ll}
M_1 = M_{(1,0,0,0,0,0;0,0,0,0,0,0)}, & M_2 = M_{(1,0,0,0,0,0;2,-1,0,0,0,0)}, \\
M_3 = M_{(0,1,0,0,0,0;0,0,0,0,0,0)}, & M_4 = M_{(0,1,0,0,0,0;-1,2,-1,0,0,0)}, \\
M_5 = M_{(0,0,1,0,0,0;0,0,0,0,0,0)}, & M_6 = M_{(0,0,1,0,0,0;0,-1,2,-1,0,-1)}, \\
M_7 = M_{(0,0,0,1,0,0;0,0,0,0,0,0)}, & M_8 = M_{(0,0,0,1,0,0;0,0,-1,2,-1,0)}, \\
M_9 = M_{(0,0,0,0,1,0;0,0,0,0,0,0)}, & M_{10} = M_{(0,0,0,0,1,0;0,0,0,-1,2,0)}, \\
M_{11} = M_{(0,0,0,0,0,1;0,0,0,0,0,0)}, & M_{12} = M_{(0,0,0,0,0,1;0,0,-1,0,0,2)},
\end{array}
\tag{3.19}
$$

and the other exact monodromies can be obtained by the conjugation of the above basic monodromies. We note that the product of any pairs of exact monodromies in (3.19) reproduce the semiclassical monodromies, i.e.

$$
\begin{array}{lll}
M_1 M_2 = M^{(r_1)}, & M_3 M_4 = M^{(r_2)}, & M_5 M_6 = M^{(r_3)} \\
M_7 M_8 = M^{(r_4)}, & M_9 M_{10} = M^{(r_5)}, & M_{11} M_{12} = M^{(r_6)}
\end{array}
\tag{3.20}
$$

III. E_7 Gauge Group

The group E_7 is of rank 7 and dimension 133 and its Dynkin diagram is



The Weyl group of E_7 is of the order 2903040, that is generated by

$$
\begin{array}{lcl}
r_1 & : & (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \rightarrow (-a_1 + a_2, a_2, a_3, a_4, a_5, a_6, a_7), \\
r_2 & : & (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \rightarrow (a_1, a_1 - a_2 + a_3, a_3, a_4, a_5, a_6, a_7), \\
r_3 & : & (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \rightarrow (a_1, a_2, a_2 - a_3 + a_4 + a_7, a_4, a_5, a_6, a_7), \\
r_4 & : & (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \rightarrow (a_1, a_2, a_3, a_3 - a_4 + a_5, a_5, a_6, a_7), \\
r_5 & : & (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \rightarrow (a_1, a_2, a_3, a_4, a_4 - a_5 + a_6, a_6, a_7), \\
r_6 & : & (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \rightarrow (a_1, a_2, a_3, a_4, a_5, a_5 - a_6, a_7), \\
r_7 & : & (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \rightarrow (a_1, a_2, a_3, a_4, a_5, a_6, a_3 - a_7).
\end{array}
\tag{3.21}
$$

We choose the fundamental representation Λ_6 which its dimension is 56, so (2.2) is

$$
\begin{array}{ll}
W & (x) = (x^2 - a_6^2)(x^2 - (a_5 - a_6)^2)(x^2 - (a_4 - a_5)^2) \\
 & (x^2 - (a_3 - a_4)^2)(x^2 - (a_2 - a_3 + a_7)^2)(x^2 - (a_1 - a_2 + a_7)^2) \\
 & (x^2 - (a_2 - a_7)^2)(x^2 - (a_7 - a_1)^2)(x^2 - (a_1 - a_2 + a_3 - a_7)^2) \\
 & (x^2 - (a_3 - a_7 - a_1)^2)(x^2 - (-a_3 + a_4 + a_1)^2)(x^2 - (-a_1 + a_2 - a_3 + a_4)^2) \\
 & (x^2 - (a_5 - a_4 + a_1)^2)(x^2 - (a_1 - a_5 + a_6)^2)(x^2 - (-a_3 + a_4 - a_6 + a_7)^2) \\
 & (x^2 - (-a_1 + a_2 - a_4 + a_5)^2)(x^2 - (-a_1 + a_2 - a_5 + a_6)^2)(x^2 - (a_1 - a_6)^2) \\
 & (x^2 - (-a_2 + a_3 - a_4 + a_5)^2)(x^2 - (-a_1 + a_2 - a_6)^2)(x^2 - (-a_3 + a_5 + a_7)^2) \\
 & (x^2 - (-a_2 + a_3 - a_5 + a_6)^2)(x^2 - (-a_2 + a_3 - a_6)^2)(x^2 - (a_2 - a_4)^2)
\end{array}
$$

$$
\begin{aligned}
&(x^2-(a_4-a_6-a_7)^2)(x^2-(a_5-a_7)^2)(x^2-(-a_4+a_5-a_6+a_7)^2)\\
&(x^2-(-a_3+a_4-a_5+a_6+a_7)^2),
\end{aligned}
\tag{3.22}
$$

and the quantum curve is

$$
y^2 = W^2(x) - \Lambda^{36}x^{76}, \tag{3.23}
$$

that can be express in terms of Casimir invariants of E_7 which are $u_2, u_6, u_8, u_{10}, u_{12}$, u_{14} and u_{18}.

Similar to F_4 and E_6 cases, the classical and quantum discreminant can be factoraized and hence we have many "singularites" which are again unphysical. The hyperelleptic curve (3.23) has 56 cuts that can be written from classical level surface. These 56 cycles vanish when $\Lambda \to 0$, so one can compute the monodromies for these singularities (B_i, $i = 1, \cdots, 28$) and obtain the quantum shift matrix $\prod_{i=1}^{28} B_i = T^{-6}$ where $T = \begin{pmatrix} \mathbb{1} & C \\ 0 & \mathbb{1} \end{pmatrix}$ and C is the Cartan matrix of E_7, in agreement with the one loop corrected prepotential. The simple semiclassical monodromies obtained from the one loop corrected prepotential are in the form (3.10). Now C_i's are

$$
C_1 = \begin{pmatrix}
-4 & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \qquad
C_2 = \begin{pmatrix}
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
2 & -4 & 2 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
C_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & -1 \\
0 & 2 & -4 & 2 & 0 & 0 & -2 \\
0 & -1 & 2 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & -1
\end{pmatrix}, \quad
C_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 2 & -4 & 2 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
C_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 2 & -4 & 2 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
C_6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$C_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -4 \end{pmatrix}. \tag{3.24}$$

The simple exact monodromies can be obtained in the same way as previous cases and are

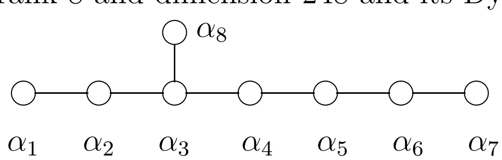
$$\begin{aligned} &M_1 = M_{(1,0,0,0,0,0,0;0,0,0,0,0,0,0)}, \quad &&M_2 = M_{(1,0,0,0,0,0,0;2,-1,0,0,0,0,0)}, \\ &M_3 = M_{(0,1,0,0,0,0,0;0,0,0,0,0,0,0)}, \quad &&M_4 = M_{(0,1,0,0,0,0,0;-1,2,-1,0,0,0,0)}, \\ &M_5 = M_{(0,0,1,0,0,0,0;0,0,0,0,0,0,0)}, \quad &&M_6 = M_{(0,0,1,0,0,0,0;0,-1,2,-1,0,0,-1)}, \\ &M_7 = M_{(0,0,0,1,0,0,0;0,0,0,0,0,0,0)}, \quad &&M_8 = M_{(0,0,0,1,0,0,0;0,0,-1,2,-1,0,0)}, \\ &M_9 = M_{(0,0,0,0,1,0,0;0,0,0,0,0,0,0)}, \quad &&M_{10} = M_{(0,0,0,0,1,0,0;0,0,0,-1,2,-1,0)}, \\ &M_{11} = M_{(0,0,0,0,0,1,0;0,0,0,0,0,0,0)}, \quad &&M_{12} = M_{(0,0,0,0,0,1,0;0,0,0,0,-1,2,0)}, \\ &M_{13} = M_{(0,0,0,0,0,0,1;0,0,0,0,0,0,0)}, \quad &&M_{14} = M_{(0,0,0,0,0,0,1;0,0,-1,0,0,0,2)}, \end{aligned} \tag{3.25}$$

and the other exact monodromies can be obtained by the conjugation of above basic monodromies. We note that the product of any pairs of exact monodromies in (3.25) reproduces the semiclassical monodromies, i.e.

$$\begin{aligned} &M_1 M_2 = M^{(r_1)}, \quad &&M_3 M_4 = M^{(r_2)}, \quad &&M_5 M_6 = M^{(r_3)}, \\ &M_7 M_8 = M^{(r_4)}, \quad &&M_9 M_{10} = M^{(r_5)}, \quad &&M_{11} M_{12} = M^{(r_6)}, \\ &M_{13} M_{14} = M^{(r_7)}. \end{aligned} \tag{3.26}$$

IV. E_8 Gauge Group

The group E_8 is of rank 8 and dimension 248 and its Dynkin diagram is



The Weyl group of E_8 is of the order 696729600, that is generated by eight simple reflections corresponding to simple roots. Because of the complexity of the results, here we only list some of them. We choose the fundamental representation Λ_7 which is 248 dimensional, so (2.2) is

$$W(x) = \prod_{i=1}^{120} (x^2 - b_i^2) \tag{3.27}$$

where b_i's are given in Appendix. The Weyl invariant quantum curve is in the following form

$$y^2 = W^2(x) - \Lambda^{60} x^{420}, \tag{3.28}$$

that can be express in terms of the Casimir invariants of E_8 which are $u_2, u_8, u_{12}, u_{14}, u_{18}, u_{20}, u_{24}, u_{30}$.

Similar to the previous exceptional gauge groups, the classical and quantum discriminants can be factoraized and we have many "singularities" which seem unphysical. The hyperelliptic curve (3.28) has 240 cuts, that half of them are related to the other half by the parity. These 120 cycles vanish when $\Lambda \to 0$, so one can compute the monodromies for these singularities (B_i, $i = 1, \cdots, 120$) and then the quantum shift matrix becomes in the form $T = \begin{pmatrix} \mathbb{1} & C \\ 0 & \mathbb{1} \end{pmatrix}$ that C is the Cartan matrix of E_8. The simple semiclassical monodromies obtained from the one loop corrected prepotential are in the form (3.10). The simple exact monodromies can be obtained in the same way as the previous cases and the results are

$$\begin{array}{ll} M_1 = M_{(1,0,0,0,0,0,0,0;0,0,0,0,0,0,0,0)}, & M_2 = M_{(1,0,0,0,0,0,0,0;2,-1,0,0,0,0,0,0)}, \\ M_3 = M_{(0,1,0,0,0,0,0,0;0,0,0,0,0,0,0,0)}, & M_4 = M_{(0,1,0,0,0,0,0,0;-1,2,-1,0,0,0,0,0)}, \\ M_5 = M_{(0,0,1,0,0,0,0,0;0,0,0,0,0,0,0,0)}, & M_6 = M_{(0,0,1,0,0,0,0,0;0,-1,2,-1,0,0,0,-1)}, \\ M_7 = M_{(0,0,0,1,0,0,0,0;0,0,0,0,0,0,0,0)}, & M_8 = M_{(0,0,0,1,0,0,0,0;0,0,-1,2,-1,0,0,0)}, \\ M_9 = M_{(0,0,0,0,1,0,0,0;0,0,0,0,0,0,0,0)}, & M_{10} = M_{(0,0,0,0,1,0,0,0;0,0,0,-1,2,-1,0,0)}, \\ M_{11} = M_{(0,0,0,0,0,1,0,0;0,0,0,0,0,0,0,0)}, & M_{12} = M_{(0,0,0,0,0,1,0,0;0,0,0,0,-1,2,-1,0)}, \\ M_{13} = M_{(0,0,0,0,0,0,1,0;0,0,0,0,0,0,0,0)}, & M_{14} = M_{(0,0,0,0,0,0,1,0;0,0,0,0,0,-1,2,0)}, \\ M_{15} = M_{(0,0,0,0,0,0,0,1;0,0,0,0,0,0,0,0)}, & M_{16} = M_{(0,0,0,0,0,0,0,1;0,0,-1,0,0,0,0,2)}, \end{array} \tag{3.29}$$

and the other exact monodromies can be obtained by the conjugation of the above basic monodromies. One can see that by multiplication of any pairs of exact monodromies in (3.29), the semiclassical monodromies are reproduced.

4 Conclusions

In this article, we have presented the hyperelliptic curves for any Lie gauge group. We have found that they have the form $y^2 = W^2(x) - \Lambda^{2\hat{h}} x^k$ where $W(x)$ is a polynomial of order n (n is the dimension of the fundamental representation with least dimension minus the number of zero weights) , $\hat{h}$ is the dual Coxeter number of the Lie gauge group and $k < 2n$. It has been found in [3] that the one form λ for these hyperelliptic curves is $\lambda = (\frac{k}{2}W - xW')\frac{dx}{y}$, and we have been found that the simple exact monodromies can be obtained only from the Cartan matrix ($C = \{C_{ij}\}$). In this basis, we have $r = rank\,(G)$ massless monopoles [2]. The simple exact monodromies are in the form $M_{(e_i,0)}$ and $M_{(e_i, \sum_{j=1}^{r} C_{ij} e_j)}$, $i = 1, \cdots, r$, for every simple root of G, where e_i is the unit vector in the ith direction of Cartan subspace. The simple semiclassical monodromies can also be obtained from the rows of the Cartan matrix. The other monodromies can be obtained by the conjugation.

It is interesting to note that by deleting some vertex in Dynkin diagram of the gauge group G, one can find the hyperelliptic curve of the related subgroup of G from the curve of G[3]. For example, by deleting the vertex α_1 or α_4 from the

Dynkin diagram of F_4, one can find the curve of $SP(6)$ or $SO(7)$ respectively from the hyperelliptic curve (3.3) of F_4. Similar results can be obtained from the other exceptional curves.

ACKNOWLEDGEMENTS

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APPENDIX

Here we list the b_i's factor appearing in the classical curve (3.27)

$$
\begin{array}{ll}
b_1 = a_7, & b_2 = a_7 - a_6, \\
b_3 = a_6 - a_5, & b_4 = a_5 - a_4, \\
b_5 = a_4 - a_3, & b_6 = a_8 - a_3 + a_2, \\
b_7 = a_8 - a_2 + a_1, & b_8 = a_8 - a_2, \\
b_9 = a_8 - a_1, & b_{10} = a_8 - a_3 + a_2 - a_1, \\
b_{11} = a_8 - a_3 + a_1, & b_{12} = a_4 - a_3 + a_1, \\
b_{13} = a_4 - a_3 + a_2 - a_1, & b_{14} = a_5 - a_4 + a_1, \\
b_{15} = a_4 - a_2, & b_{16} = a_5 - a_4 + a_2 - a_1, \\
b_{17} = a_6 - a_5 + a_1, & b_{18} = a_5 - a_4 + a_3 - a_2, \\
b_{19} = a_6 - a_5 + a_2 - a_1, & b_{20} = a_7 - a_6 + a_1, \\
b_{21} = a_8 + a_5 - a_3, & b_{22} = a_6 - a_5 + a_3 - a_2, \\
b_{23} = a_7 - a_6 + a_2 - a_1, & b_{24} = a_7 - a_1, \\
b_{25} = a_8 - a_5, & b_{26} = a_8 + a_6 - a_5 + a_4 - a_3, \\
b_{27} = a_7 - a_6 + a_3 - a_2, & b_{28} = a_7 - a_2 + a_1, \\
b_{29} = a_8 - a_6 + a_5 - a_4, & b_{30} = a_8 + a_6 - a_4, \\
b_{31} = a_8 + a_7 - a_6 + a_4 - a_3, & b_{32} = a_7 - a_3 + a_2, \\
b_{33} = a_8 - a_6 + a_4 - a_3, & b_{34} = a_8 + a_7 - a_6 + a_5 - a_4, \\
b_{35} = a_8 - a_7 + a_6 - a_4, & b_{36} = a_8 - a_7 + a_4 - a_3, \\
b_{37} = a_6 - a_3 + a_2, & b_{38} = a_8 - a_7 + a_6 - a_5 + a_4 - a_3, \\
b_{39} = a_8 + a_7 - a_5, & b_{40} = a_8 - a_7 + a_5 - a_4, \\
b_{41} = a_8 + a_7 - a_4, & b_{42} = a_6 - a_2 + a_1, \\
b_{43} = a_7 - a_6 + a_5 - a_3 + a_2, & b_{44} = a_8 + a_7 - a_5 + a_4 - a_3, \\
b_{45} = a_8 - a_7 + a_5 - a_3, & b_{46} = a_7 - a_6 + a_5 - a_2 + a_1, \\
b_{47} = a_6 - a_1, & b_{48} = a_8 - a_7 + a_6 - a_5, \\
b_{49} = a_7 - a_5 + a_3 - a_2, & b_{50} = a_7 - a_5 + a_4 - a_3 + a_2, \\
b_{51} = a_8 - a_6, & b_{52} = a_8 + a_7 - a_6 + a_5 - a_3, \\
b_{53} = a_7 - a_6 + a_5 - a_1, & b_{54} = a_7 - a_5 + a_2 - a_1, \\
b_{55} = a_7 - a_5 + a_4 - a_2 + a_1, & b_{56} = a_7 - a_4 + a_2, \\
b_{57} = a_8 + a_6 - a_3, & b_{58} = a_7 - a_6 + a_5 - a_4 + a_3 - a_2, \\
b_{59} = a_7 - a_5 + a_1, & b_{60} = a_7 - a_5 + a_4 - a_1, \\
b_{61} = a_7 - a_4 + a_3 - a_2 + a_1, & b_{62} = a_7 - a_6 + a_5 - a_4 + a_2 - a_1,
\end{array}
$$

$$
\begin{aligned}
b_{63} &= a_7 - a_6 + a_4 - a_2, & b_{64} &= a_6 - a_4 + a_3 - a_2,\\
b_{65} &= a_8 + a_7 - a_3 + a_1, & b_{66} &= a_7 - a_4 + a_3 - a_1,\\
b_{67} &= a_7 - a_6 + a_5 - a_4 + a_1, & b_{68} &= a_7 - a_6 + a_4 - a_3 + a_2 - a_1,\\
b_{69} &= a_6 - a_5 + a_4 - a_2, & b_{70} &= a_8 + a_7 - a_3 + a_2 - a_1,\\
b_{71} &= a_8 - a_7 - a_1, & b_{72} &= a_6 - a_4 + a_2 - a_1,\\
b_{73} &= a_7 - a_6 + a_4 - a_3 + a_1, & b_{74} &= a_8 - a_7 + a_6 - a_3 + a_1,\\
b_{75} &= a_5 - a_2, & b_{76} &= a_6 - a_5 + a_4 - a_3 + a_2 - a_1,\\
b_{77} &= a_6 - a_4 + a_1, & b_{78} &= a_8 - a_7 - a_2 + a_1,\\
b_{79} &= a_8 - a_7 + a_6 - a_3 + a_2 - a_1, & b_{80} &= a_8 + a_7 - a_6 - a_1,\\
b_{81} &= a_8 + a_7 - a_2, & b_{82} &= a_6 - a_5 + a_4 - a_3 + a_1,\\
b_{83} &= a_8 - a_6 + a_5 - a_3 + a_1, & b_{84} &= a_5 - a_3 + a_2 - a_1,\\
b_{85} &= a_8 - a_7 - a_3 + a_2, & b_{86} &= a_8 + a_7 - a_6 - a_2 + a_1,\\
b_{87} &= a_8 - a_7 + a_6 - a_2, & b_{88} &= a_8 - a_6 + a_5 - a_3 + a_2 - a_1,\\
b_{89} &= a_8 - a_5 + a_4 - a_3 + a_1, & b_{90} &= a_8 + a_6 - a_5 - a_1,\\
b_{91} &= a_8 + a_7 - a_6 - a_3 + a_2, & b_{92} &= a_7 + a_4 - a_3,\\
b_{93} &= a_5 - a_3 + a_1, & b_{94} &= a_8 + a_6 - a_5 - a_2 + a_1,\\
b_{95} &= a_8 - a_4 + a_1, & b_{96} &= a_8 + a_5 - a_4 - a_1,\\
b_{97} &= a_8 - a_6 + a_5 - a_2, & b_{98} &= a_8 - a_5 + a_4 - a_3 + a_2 - a_1,\\
b_{99} &= a_7 + a_5 - a_4, & b_{100} &= a_8 + a_6 - a_5 - a_3 + a_2,\\
b_{101} &= a_8 + a_4 - a_3 - a_1, & b_{102} &= a_7 - a_6 - a_4 + a_3,\\
b_{103} &= a_8 + a_5 - a_4 - a_2 + a_1, & b_{104} &= a_8 - a_4 + a_2 - a_1,\\
b_{105} &= a_8 - a_5 + a_4 - a_2, & b_{106} &= a_7 + a_6 - a_5,\\
b_{107} &= a_7 - a_6 - a_5 + a_4, & b_{108} &= a_3 - a_2 - a_1,\\
b_{109} &= a_8 + a_5 - a_4 - a_3 + a_2, & b_{110} &= a_8 + a_4 - a_3 - a_2 + a_1,\\
b_{111} &= a_6 - a_5 - a_4 + a_3, & b_{112} &= a_8 - a_4 + a_3 - a_2,\\
b_{113} &= 2a_7 - a_6, & b_{114} &= a_7 - 2a_6 + a_5,\\
b_{115} &= a_2 - 2a_1, & b_{116} &= a_6 - 2a_5 + a_4,\\
b_{117} &= a_3 - 2a_2 + a_1, & b_{118} &= a_5 - 2a_4 + a_3,\\
b_{119} &= a_8 + a_4 - 2a_3 + a_2, & b_{120} &= 2a_8 - a_3.
\end{aligned}
$$

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